

Vertex partitions of metric spaces with finite distance sets

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January 04 2010

Abstract

A metric space $M = (M, d)$ is *indivisible* if for every colouring $\chi : M \rightarrow 2$ there exists $i \in 2$ and a copy $N = (N, d)$ of M in M so that $\chi(x) = i$ for all $x \in N$. The metric space M is *homogeneous* if for every isometry α of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending α . A homogeneous metric space $U_{\mathcal{D}}$ with \mathcal{D} as set of distances is an *Urysohn metric space* if every finite metric space with set of distances a subset of \mathcal{D} has an isometry into $U_{\mathcal{D}}$. The main result of this paper states that all countable Urysohn metric spaces with a finite set of distances are indivisible.

1 Introduction

The connection between structural Ramsey theory, Fraïssé Theory and topological dynamics established in [8] and [3] leads naturally to the partition problem addressed in this paper. See [6] for a more extensive discussion and for establishing claimed facts. Detailed introductions to Fraïssé limits can be found in [2] or [9]. Below is a short account of the origin of the problem discussed in this paper.

For $(M; d)$ a metric space let $\text{dist}(M; d)$ be the set of distances between points of $(M; d)$. A metric space M is *homogeneous* if for every isometry α of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending α . A homogeneous metric space $\mathbf{U}_{\mathcal{D}}$ with $\text{dist}(\mathbf{U}_{\mathcal{D}}) = \mathcal{D}$ is an *Urysohn metric space* if every finite metric space M with $\text{dist}(M) \subseteq \mathcal{D}$ has an isometry into $\mathbf{U}_{\mathcal{D}}$. As Urysohn metric spaces are universal objects, a subset $\mathcal{D} \subseteq \mathbb{R}_{\geq 0}$ is called *universal* if there exists an Urysohn metric space $\mathbf{U}_{\mathcal{D}}$. To decide whether a given set $\mathcal{D} \subseteq \mathbb{R}_{\geq 0}$ is universal can be difficult. A particular example of an Urysohn space is the *Urysohn sphere* $\mathbf{U}_{\mathbb{R} \cap [0,1]}$. Other examples are the Urysohn spaces \mathbf{U}_m with $m \in \omega$ for which $\text{dist}(\mathbf{U}_m)$ is equal to $\{0, 1, 2, \dots, m-1\}$ and the Hilbert space ℓ_2 .

A *copy* of a metric space $(M; d)$ in $(M; d)$ is the image of an isometry of $(M; d)$ in $(M; d)$. The space $M = (M; d)$ is *indivisible* if for every colouring $\chi : M \rightarrow 2$ there exists $i \in 2$ and a copy M^* of M in M so that $\chi(x) = i$ for all $x \in M^*$.

A metric space $(M; d)$ is *oscillation stable* if for every bounded and uniformly continuous function $f : M \rightarrow \mathbb{R}$ and every $\epsilon > 0$ there is a copy (M^*, d) of $(M; d)$ in $(M; d)$ so that:

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

The question whether the unit sphere of the Hilbert space ℓ_2 is oscillation stable had been known as the *distortion problem* and was finally resolved in the negative, see [7]. This then led to the question whether the other prominent bounded metric space with a large isometry group, namely the Urysohn sphere $\mathbf{U}_{\mathbb{R} \cap [0,1]}$, is oscillation stable. After an initial reformulation of the problem by V. Pestov, see [8], Lopez-Abad and Nguyen Van Thé, see [4], started a programme to reduce the problem to one of discrete mathematics. In particular they proved that the Urysohn sphere will be oscillation stable if and only if each of the Urysohn spaces \mathbf{U}_m is indivisible. Subsequently it was established in [5] that all of the Urysohn spaces \mathbf{U}_m are indivisible, finishing the proof that the Urysohn sphere $\mathbf{U}_{\mathbb{R} \cap [0,1]}$ is oscillation stable.

Which metric spaces are oscillation stable? There does not seem to be any way at present to attack this question in general. Even to ask for a characterization of the oscillation stable homogeneous metric spaces is beyond our present means. But, to find a characterization of the oscillation stable Urysohn metric spaces might just be possible following the ideas of Lopez-Abad and Nguyen Van Thé to reformulate the problem as a problem of discrete mathematics. There are essentially two steps to such a characterization. Step 1 is to investigate whether

the Urysohn metric spaces $\mathbf{U}_{\mathcal{D}}$ for \mathcal{D} finite are indivisible. An, as I think, attractive question in its own right, belonging to the general area of structural Ramsey theory. The present paper contains the proof that all Urysohn metric spaces $\mathbf{U}_{\mathcal{D}}$ with \mathcal{D} finite are indivisible, see Theorem 9.1.

Step 2 is to establish a connection between the oscillation stability of Urysohn spaces $\mathbf{U}_{\mathcal{D}}$ for general bounded \mathcal{D} and the indivisibility of the ones for finite \mathcal{D} , following [4]. This connection between the two problems is to appear in a forthcoming paper. If \mathcal{D} is not bounded $\mathbf{U}_{\mathcal{D}}$ can not be oscillation stable. (Unpublished, but generally known in the area, the proof being an easy modification of the proof of Theorem 3.14 in [1].)

The objects discussed in this paper are metric spaces with a finite set of distances. They may be viewed as labelled graphs or relational structures. Relational structures are usually denoted by the same letter as their base sets, just using a different font. On the other hand, metric spaces carry a natural topology and topological spaces are often just denoted by their base sets. Here $M = (M; d_M)$ is the full description of a metric space, used mostly if different metrics on M are needed. This description will often be abbreviated to $(M; d)$ or just M if the metric is given by context.

2 Notation and basic facts

Let $0 \in \mathcal{D} \subseteq \mathfrak{R}_{\geq 0}$ be a given finite set of numbers.

A pair $H = (H, d)$ is a \mathcal{D} -graph if $d : H^2 \rightarrow \mathcal{D}$ is a function with $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in H$. For $A \subseteq H$ we denote by $H \upharpoonright A$ the *substructure of H generated by A* , that is the \mathcal{D} -graph on A with distance function the restriction of d to A^2 . The \mathcal{D} -graph H is *metric* if it is a metric space. That is if $d(p, q) \leq d(p, r) + d(r, q)$ for all triples $(p, q, r) \in H^3$. Let $\mathfrak{M}_{\mathcal{D}}$ be the class of metric spaces M with $\text{dist}(M) \subseteq \mathcal{D}$ and $\mathfrak{U}_{\mathcal{D}}$ the class of countable Urysohn metric spaces in $\mathfrak{M}_{\mathcal{D}}$. The following 4-values condition, see [6] or [1] provides a characterization of universal sets of numbers.

Lemma 2.1. *The set \mathcal{D} of numbers is universal if and only if for any two triangles a_0, b, c and a_1, b, c which are in $\mathfrak{M}_{\mathcal{D}}$, there exists $0 < t \in \mathcal{D}$ so that the \mathcal{D} -graph $L = (\{a_0, a_1, b, c\}, d)$ with $d(a, a_1) = t$ and for which the two triangles are induced subspaces, is an element of $\mathfrak{M}_{\mathcal{D}}$. (The case $d(b, c) = 0$, that is $b = c$ is included.)*

Let \mathcal{D} be a universal set of numbers. By scaling \mathcal{D} to $t\mathcal{D}$ for some positive real t the set of distances $t\mathcal{D}$ is a universal set of numbers. The metric spaces in $\mathfrak{U}_{\mathcal{D}}$ are indivisible if and only if the metric spaces in $\mathfrak{U}_{t\mathcal{D}}$ are indivisible. Hence we may assume that $\min(\mathcal{D} \setminus \{0\}) = 1$.

A function $\mathfrak{t} : F \rightarrow \mathcal{D}$, with F a finite subset of H , is a *type function* of H . For \mathfrak{t} a type function let $\text{Sp}(\mathfrak{t})$ be the \mathcal{D} -graph on $F \cup \{\mathfrak{t}\}$ for which:

1. $\text{Sp}(\mathfrak{t}) \upharpoonright \text{dom}(\mathfrak{t}) = H \upharpoonright \text{dom}(\mathfrak{t})$.
2. $\forall x \in F \ (d(\mathfrak{t}, x) = d(x, \mathfrak{t}) = \mathfrak{t}(x))$.

For \mathbf{t} a type function of H let

$$\text{orb}(\mathbf{t}) = \{y \in H \setminus \text{dom}(\mathbf{t}) : \forall x \in \text{dom}(\mathbf{t}) (d(y, x) = d(\mathbf{t}, x) = \mathbf{t}(x))\},$$

the *orbit* of \mathbf{t} . If the distinction is necessary we will write $\text{orb}_H(\mathbf{t})$. Note that if $\text{dom}(\mathbf{t}) = \emptyset$ then $\text{orb}(\mathbf{t}) = H$. If $\text{orb} \mathbf{t} \neq \emptyset$ then \mathbf{t} can be realized in H and every $p \in \text{orb} \mathbf{t}$ is a *realization* of \mathbf{t} . A type function \mathbf{t} of H is a *Katětov function* if $\text{Sp}(\mathbf{t})$ is metric. Note here that \mathcal{D} is given. Hence $\text{dist}(\text{Sp}(\mathbf{t})) \subseteq \mathcal{D}$ is assumed. The *rank* of \mathbf{t} , $\text{rank}(\mathbf{t})$, is $\min\{\mathbf{t}(x) : x \in \text{dom}(\mathbf{t})\}$.

Let $M \in \mathfrak{M}_{\mathcal{D}}$. Then a type function \mathbf{t} of M is a Katětov function if and only if for all $x, y \in \text{dom}(\mathbf{t})$:

$$|\mathbf{t}(x) - \mathbf{t}(y)| \leq d(x, y) \leq \mathbf{t}(x) + \mathbf{t}(y). \quad (1)$$

Remember $\mathbf{t}(x) \in \mathcal{D}$ for all $x \in \text{dom}(\mathbf{t})$. Because $\text{dom}(\mathbf{t}) \subseteq M$ all triangles in $\text{dom}(\mathbf{t})$ are metric. Hence, in order to check that $\text{Sp}(\mathbf{t})$ is metric it suffices to check all triangles of the form $\{\mathbf{t}, x, y\}$ with $x, y \in \text{dom}(\mathbf{t})$, which is an easy consequence of Inequalities (1).

The following Theorem 2.1 is a direct consequence of the general theory of Fraïssé limits see [6] for a more extensive discussion or [2] and [9]. The other assertions in the remainder of this section are known facts as well. Their proofs are given.

Theorem 2.1. *A countable metric space $M \in \mathfrak{M}_{\mathcal{D}}$ is an Urysohn space if and only if every Katětov function \mathbf{t} of M is realized in M . Any two Urysohn spaces in $\mathfrak{U}_{\mathcal{D}}$ are isometric, indeed:*

Let $M, N \in \mathfrak{U}_{\mathcal{D}}$, then every isometry of a finite subspace of M to a finite subspace of N can be extended to an isometry of M to N .

Let $M \in \mathfrak{U}_{\mathcal{D}}$ and $N \in \mathfrak{M}_{\mathcal{D}}$ countable. Then every isometry of a finite subspace of N into M can be extended to an isometry of N into M .

The last assertion being known as the *mapping extension property*. For \mathcal{D} universal let $\mathbf{U}_{\mathcal{D}}$ be a particular Urysohn space in the class $\mathfrak{U}_{\mathcal{D}}$.

Corollary 2.1. *Let $H = (H, d) \in \mathfrak{U}_{\mathcal{D}}$ and $C \subseteq H$. Then $H \upharpoonright C$ is a copy of H in H if and only if $\text{orb}(\mathbf{t}) \cap C \neq \emptyset$ for every Katětov function \mathbf{t} of H with $\text{dom}(\mathbf{t}) \subseteq C$.*

Theorem 2.2. *Let $H = (H, d) \in \mathfrak{U}_{\mathcal{D}}$ and \mathbf{t} be a Katětov function of H and let $\mathcal{D}_{\mathbf{t}} = \{n \in \mathcal{D} : n \leq 2 \cdot \text{rank}(\mathbf{t})\}$. Then the restriction of H to $\text{orb}(\mathbf{t})$ is isometric to $\mathbf{U}_{\mathcal{D}_{\mathbf{t}}}$. (It follows that $\text{orb}(\mathbf{t})$ is infinite.)*

Proof. Let \mathbf{f} be a type function with $\text{dom}(\mathbf{f}) \subseteq \text{orb}(\mathbf{t})$ and $\text{Sp}(\mathbf{f})$ metric and $\text{dist}(\text{Sp}(\mathbf{f})) \subseteq \mathcal{D}_{\mathbf{t}}$. Let $x \in \text{orb}(\mathbf{t})$ and \mathbf{g} the type function with $\text{dom}(\mathbf{g}) = \text{dom}(\mathbf{f}) \cup \text{dom}(\mathbf{t})$ and with $\mathbf{f} \subseteq \mathbf{g}$. For every $p \in \text{orb}(\mathbf{t})$ let $\mathbf{g}(p) = \mathbf{t}(x)$. In order to check that \mathbf{g} is metric, given that \mathbf{t} and \mathbf{f} are metric, triangles of the form $\{\mathbf{g}, a, b\}$ with $a \in \text{dom}(\mathbf{f})$ and $b \in \text{dom}(\mathbf{t})$ have to be checked to be metric. Note that $d(\mathbf{g}, b) = d(a, b)$. Hence the triangle is metric because $d(a, b) \geq \text{rank}(\mathbf{t})$ and $d(\mathbf{g}, a) \leq 2 \cdot \text{rank}(\mathbf{t})$.

It follows that \mathbf{g} is a Katětov function and hence has a realization q according to Theorem 2.1. Then $q \in \text{orb}(\mathbf{t})$ because $\mathbf{t} \subseteq \mathbf{g}$. Using Corollary 2.1 with $\mathcal{D}_{\mathbf{t}}$ for \mathcal{D} , we conclude that $\text{orb}(\mathbf{t})$ is isomorphic to $U_{\mathcal{D}_{\mathbf{t}}}$. \square

Corollary 2.2. *Let \mathbf{t} be a Katětov function of $U_{\mathcal{D}}$ with $\text{rank}(\mathbf{t}) = r$ and $s \leq 2r \in \mathcal{D}$ and $x \in \text{orb}(\mathbf{t})$. Then the type function \mathbf{p} with $\text{dom}(\mathbf{p}) = \text{dom}(\mathbf{t}) \cup \{x\}$ and $\mathbf{t} \subseteq \mathbf{p}$ and $\mathbf{p}(x) = s$ is a Katětov function.*

Lemma 2.2. *Let $M = (M, d)$ be isometric to $U_{\mathcal{D}}$ and A and B finite subsets of M with $A \cap B = \emptyset$. Then there exists an isomorphism α of M to $M \upharpoonright (M \setminus B)$ with $\alpha(a) = a$ for all $a \in A$.*

Proof. Let $M \upharpoonright (M \setminus B) = N$ and \mathbf{t} a Katětov function of N and hence a Katětov function of M . Because $\text{orb}_M(\mathbf{t})$ is infinite there is a $y \in \text{orb}(\mathbf{t}) \setminus A$. Hence \mathbf{t} is realized in N , implying $M \upharpoonright N$ is isometric to M according to Corollary 2.1. It follows from Theorem 2.1 that the identity map on A has an extension to an isometry of M to $M \upharpoonright N$. \square

3 The structure of $U_{\mathcal{D}}$

Let \mathcal{D} be a universal set of numbers and $M = (M; d) \in \mathfrak{U}_{\mathcal{D}}$.

For r a positive real let $r^{(-)} = \max([0, r) \cap \mathcal{D})$, the largest number in \mathcal{D} smaller than r . For $r < \max \mathcal{D}$ let $r^{(+)} = \min((r, \max \mathcal{D}] \cap \mathcal{D})$, the smallest number in \mathcal{D} larger than r . For $r = \max \mathcal{D}$ let $r^{(+)} = r$. The number $r \in \mathcal{D}$ is a *jump number* if $r^{(+)} > 2 \cdot r$.

Lemma 3.1. *Let $0 < m \in \mathcal{D}$ so that the set S of numbers $r \in \mathcal{D}$ with $r^{(+)} > m + r$ is not empty. Then $\min S$ is a jump number.*

Proof. Let $r = \min S$. If $m \geq r$ then $r^{(+)} > m + r$ implies $r^{(+)} > r + r$. Let $m < r$ and assume for a contradiction that $r^{(+)} \leq r + r$. Let $s \in \mathcal{D}$ be minimal with $r^{(+)} \leq s + r$. Then $s \leq r$ and $0 < s$ hence

$$0 < m \leq s^{(-)} < s \leq r < m + r \leq s^{(-)} + r < r^{(+)} \leq s + r \leq r + r.$$

It follows that the largest number in \mathcal{D} less than or equal to $s^{(-)} + r$ is r . Because $s^{(-)} < r$ it follows from the minimality of r that $s = (s^{(-)})^{(+)} \leq m + s^{(-)}$. Let $A = \{a, b, c\}$ and $A' = \{a', b, c\}$ be two triangles with $d(a, c) = s^{(-)}$ and $d(b, c) = s$ and $d(a', c) = r$ and $d(a, b) = m$ and $d(a', b) = r^{(+)}$. Then $A, A' \in \mathfrak{M}_{\mathcal{D}}$.

If there is a number $t \in \mathcal{D}$ so that the space a, b, c, a' with $d(a, a') = t$ is an element of $\mathfrak{M}_{\mathcal{D}}$ then $t \leq r$ because the triangle a, a', c is metric. On the other hand the triangle a, a', b is metric and hence $r^{(+)} \leq t + m \leq r + m < r^{(+)}$. Hence there is no such number t and it follows from Lemma 2.1 that \mathcal{D} is not universal. \square

Definition 3.1. *The set $\mathcal{B} \subseteq \mathcal{D}$ is a block of \mathcal{D} if there exists an enumeration $(b_i : i \in n + 1)$ of \mathcal{B} so that:*

1. $0 < b_i < b_{i+1}$ for all $i \in n$.
2. $b_0 > b_0^{(-)} + b_0^{(-)}$.
3. $b_{i+1} = b_i^{(+)}$ for all $i \in n$.
4. $b_i + b_0 \geq b_{i+1}$ for all $i \in n$.

Lemma 3.1 implies:

Theorem 3.1. *The distance set \mathcal{D} of a universal metric space is the union of disjoint blocks \mathcal{B}_i so that:*

1. $x < y$ for all $x \in \mathcal{B}_i$ and $y \in \mathcal{B}_{i+1}$.
2. $2 \cdot \max \mathcal{B}_i < \min \mathcal{B}_{i+1}$.
3. $x + \min(\mathcal{B}_i) \geq x^{(+)}$ for all $x \in \mathcal{B}_i$.

Let r be the maximum of a block \mathcal{B} of \mathcal{D} , that is a jump number.

It follows that the relation \sim^r on M with $x \sim^r y$ iff $d(x, y) \leq r$ is an equivalence relation and that every automorphism of M maps elements of M/\sim^r onto elements of M/\sim^r . Let $A, B, C \in M/\sim^r$ with $A \neq C \neq B$ and $a \in A$ and $b \in B$ and $c \in C$. Then there exists an automorphism α of M to M mapping a to b but $\alpha(c) \notin B$ because $d(a, c) > r$ and $d(b, x) \leq r$ for all $x \in B$. If $x \in A$ then $d(a, x) \leq r$, hence $d(b, \alpha(x)) \leq r$, hence $\alpha(x) \in B$, hence $M \upharpoonright A$ is isometric to $M \upharpoonright B$. Let \mathbf{p} be a Katětov function of M with $\text{dom}(\mathbf{p}) \subseteq A$ and $\mathbf{p}(x) \leq r$ for all $x \in \text{dom}(\mathbf{p})$. Then $\text{orb}(\mathbf{p}) \subseteq A$. Hence $M \upharpoonright A$ is a homogeneous metric space with $[0, r] \cap \mathcal{D}$ as set of distances. For $x \in M$ let $[x]_r$ denote the \sim^r equivalence class containing x .

Definition 3.2. *For $A, B \in M/\sim^r$ let*
 $d(A, B) = \{d(a, b) : a \in A \text{ and } b \in B\}$ *and let*
 $d_{\min}(A, B) = \min d(A, B)$ *and* $d_{\max}(A, B) = \max d(A, B)$.

Lemma 3.2. *Let $A, B \in M/\sim^r$ with $A \neq B$ and $a \in A$ and $n \in d(A, B)$, then:*

1. $n > 2r$.
2. *There is $b \in B$ with $d(a, b) = n$.*
3. $d_{\max}(A, B) - d_{\min}(A, B) \leq r$.

Proof. Let $a \in A$ and $b \in B$ with $d(a, b) = n \leq 2r$ and \mathbf{k} the Katětov function with $\text{dom}(\mathbf{k}) = \{a, b\}$ and $\mathbf{k}(a) = r$ and $\mathbf{k}(b) = r$. Let $c \in \text{orb}(\mathbf{k})$. Then $c \in A$ and $c \in B$, hence $A = B$.

Let $a' \in A$ and $b' \in B$ with $d(a', b') = n$ and \mathbf{k} the type function with $\text{dom}(\mathbf{k}) = \{a', b', a\}$ and $\mathbf{k}(a) = n$ and $\mathbf{k}(a') = d(a, b')$ and $\mathbf{k}(b') = d(a, a')$. Because $\text{Sp}(\mathbf{k})$ is metric the type function \mathbf{k} is a Katětov function and hence

there is an element $b \in \text{orb}(\mathfrak{k})$. Because $d(b, b') < r$ the point b is an element of B and $d(a, b) = \mathfrak{k}(a) = n$.

Let $a \in A$ and $b \in B$ with $d(a, b) = d_{\max}(A, B)$. There is $c \in B$ with $d(a, c) = d_{\min}(A, B)$. The triangle a, b, c is metric with $d(b, c) \leq r$. \square

Lemma 3.3. *The triangle A, B, C is metric for all $A, B, C \in M/\overset{r}{\sim}$ under the distance function d_{\min} .*

Proof. Let $a \in A$. According to Lemma 3.2 there exists $b \in B$ and $c \in C$ with $d(a, b) = d_{\min}(A, B)$ and $d(a, c) = d_{\min}(A, C)$. Hence $d(a, b) + d(a, c) \geq d(b, c) \geq d_{\min}(B, C)$. \square

Lemma 3.4. *Let $(A_i; i \in n \in \omega)$ be pairwise different $\overset{r}{\sim}$ -equivalence classes. Then there exist points $(a_i; i \in n)$ with $a_i \in A_i$ and $d(a_i, a_j) = d_{\min}(A_i, A_j)$ for all $i, j \in n$.*

Proof. By induction on n . Let $(A_i; i \in n \in \omega)$ be pairwise different $\overset{r}{\sim}$ -equivalence classes and $(a_i; i \in n)$ points with $a_i \in A_i$ and $d(a_i, a_j) = d_{\min}(A_i, A_j)$ for all $i, j \in n$. Let A be an $\overset{r}{\sim}$ -equivalence class different from the A_i . Let $b \in A$ and \mathfrak{p} the type function with $\text{dom}(\mathfrak{p}) = \{a_i : i \in n\} \cup \{b\}$ and $\mathfrak{p}(a_i) = d_{\min}(A_i, A)$ and $\mathfrak{p}(b) = r$. Using inequalities (1) of Section 2 it follows from Lemma 3.3 that it suffices to check the inequalities $|\mathfrak{p}(a_i) - r| \leq d(a_i, a) \leq \mathfrak{p}(a_i) + r$, in order to verify that \mathfrak{p} is a Katětov function. That is the inequalities $d_{\min}(A_i, A) - r \leq d(a_i, a) \leq d_{\min}(A_i, A) + r$ which follow from Lemma 3.2 Item (3). \square

4 Distance between orbits

Let \mathcal{D} be a universal set of numbers and \mathcal{B} a block of \mathcal{D} with $\min \mathcal{B} = \mathbf{m}$. Let $M = (M, d) \in \mathcal{U}_{\mathcal{D}}$.

Definition 4.1. *Let \mathfrak{s} and \mathfrak{t} be two Katětov functions of M . Then*

$$\begin{aligned} d(\mathfrak{s}, \mathfrak{t}) &= \{m \in \mathcal{D} : \exists x \in \text{orb}(\mathfrak{s}) \exists y \in \text{orb}(\mathfrak{t}) (d(x, y) = m)\}, \\ d_{\min}(\mathfrak{s}, \mathfrak{t}) &= \min(d(\mathfrak{s}, \mathfrak{t})), \\ d_{\max}(\mathfrak{s}, \mathfrak{t}) &= \max(d(\mathfrak{s}, \mathfrak{t})). \end{aligned}$$

Lemma 4.1. *Let A be a finite subset of M and \mathfrak{s} and \mathfrak{t} two Katětov functions with $\text{dom}(\mathfrak{s}) = \text{dom}(\mathfrak{t}) = A$. Then*

$$\begin{aligned} d(\mathfrak{s}, \mathfrak{t}) &= \\ &= \{m \in \mathcal{D} : \max\{|\mathfrak{s}(x) - \mathfrak{t}(x)| : x \in A\} \leq m \leq \min\{\mathfrak{s}(x) + \mathfrak{t}(x) : x \in A\}\}. \end{aligned}$$

Proof. Let $v \in \text{orb}(\mathfrak{s})$. The type function \mathfrak{t}' with $\text{dom}(\mathfrak{t}') = A \cup \{v\}$ and $\mathfrak{t} \subseteq \mathfrak{t}'$ and $\mathfrak{t}'(v) = m$ is a Katětov function if and only if $m \in d(\mathfrak{s}, \mathfrak{t})$. Hence, in order to check that \mathfrak{t}' is a Katětov function, we have to verify that every triangle x, v, \mathfrak{t}' with $x \in A$ is metric, that is if $|d(x, \mathfrak{t}') - d(x, v)| \leq m \leq d(x, \mathfrak{t}') + d(x, v)$. \square

Corollary 4.1. *Let A be a finite subset of M and \mathfrak{s} and \mathfrak{t} two Katětov functions with $\text{dom}(\mathfrak{s}) = \text{dom}(\mathfrak{t}) = A$. Then:*

1. $d_{\min}(\mathfrak{s}, \mathfrak{t}) = \min \{n \in \mathcal{D} : n \geq \max\{|\mathfrak{s}(x) - \mathfrak{t}(x)| : x \in A\}\}.$
2. $d_{\max}(\mathfrak{s}, \mathfrak{t}) = \max \{n \in \mathcal{D} : n \leq \min\{\mathfrak{s}(x) + \mathfrak{t}(x) : x \in A\}\}.$
3. *Let $r \in \mathcal{B}$ with $\mathbf{m} \leq r^{(-)} < r$ and $\text{rank}(\mathfrak{s}) \geq r^{(-)}$ and $\text{rank}(\mathfrak{t}) \geq \mathbf{m}$. Then $d_{\max}(\mathfrak{s}, \mathfrak{t}) \geq r$.*
4. $d_{\max}(\mathfrak{s}, \mathfrak{s}) = \max \{n \in \mathcal{D} : n \leq 2 \cdot \text{rank}(\mathfrak{s})\}.$ (Hence $d_{\max}(\mathfrak{s}, \mathfrak{s}) = d_{\max}(\mathfrak{t}, \mathfrak{t})$ iff $\text{rank}(\mathfrak{s}) = \text{rank}(\mathfrak{t})$.)
5. $d(\mathfrak{s}, \mathfrak{s}) = \{n \in \mathcal{D} : 0 \leq n \leq 2 \cdot \text{rank}(\mathfrak{s})\}.$

Proof. Item 3.: For $\text{rank}(\mathfrak{s}) \geq r$ Item 3. follows because Item 2. implies $d_{\max}(\mathfrak{s}, \mathfrak{t}) \geq \text{rank}(\mathfrak{s})$. Let $\text{rank}(\mathfrak{s}) = r^{(-)}$. Then for all $x \in A$: $\mathfrak{s}(x) + \mathfrak{t}(x) \geq \text{rank}(\mathfrak{s}) + \mathbf{m} \geq r^{(-)} + \mathbf{m} \geq r$. ($r^{(-)} + \mathbf{m} \geq r$ follows from Theorem 3.1.) \square

Lemma 4.2. *Let A be a finite subset of M and $s \in \omega$ and \mathfrak{t}_i a Katětov function with $\text{dom}(\mathfrak{t}_i) = A$ for every $i \in s$. (The Katětov functions \mathfrak{t}_i need not be pairwise different.) Let d'' be a metric on the index set s so that $d''(i, j) \in d(\mathfrak{t}_i, \mathfrak{t}_j)$ for all $i, j \in s$. Then the distance function d' on $A \cup s$ given by*

1. $d'(x, y) = d(x, y)$ for all $x, y \in A$ and $d'(i, j) = d''(i, j)$ for all $i, j \in s$,
2. $d'(x, i) = \mathfrak{t}_i(x)$ for all $x \in A$ and $i \in s$,

is metric and every partial isometry β of $(A \cup C, d')$ for $C \subseteq s$ into M with $\beta(x) = x$ for all $x \in A$ has an extension to an isometry α of $(A \cup s; d')$ into M with $\alpha(i) \in \text{orb}(\mathfrak{t}_i)$ for all $i \in s$.

Proof. Because d'' is a metric on s and d is a metric on A and every \mathfrak{t}_i is a Katětov function it remains to check that the triangles of the form x, i, j with $x \in A$ and $i, j \in s$ are metric in the distance function d' . Which indeed is the case because $|d'(x, i) - d'(x, j)| = |\mathfrak{t}_i(x) - \mathfrak{t}_j(x)| \leq d_{\min}(\mathfrak{t}_i, \mathfrak{t}_j) \leq d''(i, j) = d'(i, j) = d''(i, j) \leq d_{\max}(\mathfrak{t}_i, \mathfrak{t}_j) \leq \mathfrak{t}_i(x) + \mathfrak{t}_j(x) = d'(x, i) + d'(x, j)$.

It follows from the mapping extension property that every partial isometry β of $A \cup C$ for $C \subseteq s$ into M with $\beta(x) = x$ for all $x \in A$ has an extension to an isometry α of $(A \cup s; d')$ into M . Item 2. implies that $\alpha(i) \in \text{orb}(\mathfrak{t}_i)$. \square

Corollary 4.2. *Given the conditions of Lemma 4.2: For every $k \in s$ and $v \in \text{orb}(\mathfrak{t}_k)$ there exists a set of points $\{w_i : i \in s\}$ so that $w_k = v$ and $w_i \in \text{orb}(\mathfrak{t}_i)$ and $d(w_i, w_j) = d''(i, j)$ for all $i, j \in s$.*

Lemma 4.3. *Let A, B, R be finite disjoint subsets of M and \mathfrak{T} a set of Katětov functions \mathfrak{t} with $\text{dom}(\mathfrak{t}) = A$. For every $\mathfrak{t} \in \mathfrak{T}$ let \mathfrak{t}' be a Katětov function with $\mathfrak{t} \subseteq \mathfrak{t}'$ and $\text{dom}(\mathfrak{t}') = A \cup B$. Also: $d(\mathfrak{t}, \mathfrak{s}) = d(\mathfrak{t}', \mathfrak{s}')$ for all $\mathfrak{t}, \mathfrak{s} \in \mathfrak{T}$. (Implying $\text{rank}(\mathfrak{t}) = \text{rank}(\mathfrak{t}')$ for all $\mathfrak{t} \in \mathfrak{T}$.)*

Then there exists an embedding α of $A \cup R$ into M so that $\alpha(x) = x$ for all $x \in A$ and $\alpha(y) \in \text{orb}(\mathfrak{t}')$ for all $\mathfrak{t} \in \mathfrak{T}$ and all $y \in \text{orb}(\mathfrak{t}) \cap R$.

Proof. Let $S = \{x \in R : \exists t \in \mathfrak{T} (x \in \text{orb}(t))\}$. Then $x \notin \text{orb}(\mathfrak{s})$, if $x \in \text{orb}(t)$ and $t \neq \mathfrak{s}$. Let $(s_i; i \in n)$ be an enumeration of the elements of S . For every $i \in n$ let $t_i \in \mathfrak{T}$ be such that $s_i \in \text{orb}(t_i)$. (Which implies that a $t \in \mathfrak{T}$ might be enumerated several times.) Let d'' be the metric on n with $d''(i, j) = d(s_i, s_j)$. Then $d''(i, j) \in d(t_i, t_j) = d(t'_i, t'_j)$ and hence it follows from Lemma 4.2 that there exists an embedding β of $A \cup B \cup n$ into M with $\beta(x) = x$ for all $x \in A \cup B$ and $\beta(i) \in t'_i$ for all $i \in n$. This in turn implies that there is an embedding γ of $A \cup B \cup S$ into M with $\gamma(x) = x$ for all $x \in A \cup B$ and $\gamma(s_i) \in t'_i$ for all $i \in n$.

Let α be the extension of γ to $A \cup B \cup R$. \square

5 The orbit amalgamation theorem

Let \mathcal{D} be a universal set of numbers and $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$. Let $E = (v_k; k \in \omega)$ be an ω -enumeration of M . For $m \in \omega$ we denote by $E_m = (v_k; k \in m)$ the initial interval of E of length m .

Lemma 5.1. *Let $l \in m \in \omega$ and $s \in \omega$. For all $i, j \in s$ let $t_i \subseteq s_i$ be Katětov functions with $\text{dom}(t_i) = E_l$ and $\text{dom}(s_i) = E_m$ and $d(t_i, t_j) = d(s_i, s_j)$. (Hence $\text{rank}(t_i) = \text{rank}(s_i)$).*

Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in E_l$ and $\alpha(x) \in \text{orb}(s_i)$ for all $x \in \text{orb}(t_i)$.

Proof. Let $M' = (M \setminus E_m) \cup E_l$. According to Lemma 2.2 it suffices to prove that there exists an embedding α of $M \upharpoonright M'$ into $M \upharpoonright M'$ with $\alpha(x) = x$ for all $x \in E_l$ and $\alpha(x) \in \text{orb}(s_i)$ for all $x \in \text{orb}(t_i) \cap M'$.

For $m \leq h \in \omega$ let \mathcal{A}_h be the set of isometries β with $\beta(v_k) = v_k$ for all $k \in l$ and $\text{dom}(\beta) = (E_h \setminus E_m) \cup E_l$ and $\beta(x) \in \text{orb}(s_i)$ for all $x \in \text{orb}(t_i)$ and $i \in s$. It follows from Lemma 4.3 that \mathcal{A}_j is not empty. Let $\mathcal{A} = \bigcup_{m \leq h \in \omega} \mathcal{A}_h$. For two isometries β and γ in \mathcal{A} let $\beta \preceq \gamma$ if $\beta \in \mathcal{A}_j$ and $\gamma \in \mathcal{A}_h$ with $j \leq h$ and if $d(y, \beta(x)) = d(y, \gamma(x))$ for all $y \in E_m \setminus E_l$ and all $x \in \text{dom}(\beta)$. Let $\beta \sim \gamma$ if $\beta \preceq \gamma$ and $\gamma \preceq \beta$. Then \sim is an equivalence relation and because \mathcal{D} is finite there are only finitely many \sim equivalence classes in every \mathcal{A}_h . The quasiorder \preceq on \mathcal{A} factors into a partial order \mathfrak{P} on the \sim equivalence classes. The restriction of an isometry $\beta \in \mathcal{A}_{h+1}$ to E_h is an isometry in \mathcal{A}_h .

According to König's Lemma there exists a chain $(\mathcal{C}_{m+j}; j \in \omega)$ in \mathfrak{P} . (\mathcal{C}_m being the equivalence class containing the empty function.) The Theorem will follow if for every h with $m \leq h \in \omega$ and $\beta \in \mathcal{C}_h$ there is a $\gamma \in \mathcal{C}_{h+1}$ with $\beta \preceq \gamma$.

Let $\beta \in \mathcal{C}_h$ and $\delta \in \mathcal{C}_{h+1}$. Let κ be the isometry with $\text{dom}(\kappa) = \{\delta(v_k) : k \in h\}$ for which $\kappa(\delta(v_k)) = \beta(v_k)$ for all $k \in h \setminus m$ and $\kappa(v_k) = v_k$ for all $k \in m$. Let μ be an extension of κ to an isometry of $\{\delta(v_k) : k \in h+1\}$ into M . That is the domain of μ contains the element v_h in addition to the elements in the domain of κ . Such an extension exists because M is homogeneous. Then $\beta \preceq \gamma := \beta \cup \{v_h, \mu(\delta(v_h))\}$. It remains to show that $\gamma \in \mathcal{C}_{h+1}$.

Let $i \in s$ and $v_h \in \text{orb}(t_i)$. Then $\delta(v_h) \in \text{orb}(s_i)$ because $\delta \in \mathcal{C}_{h+1} \subseteq \mathcal{A}$ implying that $\mu(\delta(v_h)) \in \text{orb}(s_i)$ because $\mu(x) = x$ for all $x \in E_m = \text{dom}(s_i)$. It follows that $\gamma \in \mathcal{C}_{h+1}$. \square

Theorem 5.1. *Let A and B be finite subsets of M with $A \cap B = \emptyset$ and $s \in \omega$ and for every $i \in s$ let $\mathbf{t}_i \subseteq \mathbf{s}_i$ be Katětov functions with $\text{dom}(\mathbf{t}_i) = A$ and $\text{dom}(\mathbf{s}_i) = A \cup B$ and $d(\mathbf{t}_i, \mathbf{t}_j) = d(\mathbf{s}_i, \mathbf{s}_j)$ for all $i, j \in s$.*

Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in A$ and $\alpha(x) \in \text{orb}(\mathbf{s}_i)$ for all $x \in \text{orb}(\mathbf{t}_i)$ and all $i \in s$; that is with $\text{orb}_{\alpha(M)}(\mathbf{t}_i) \subseteq \text{orb}_M(\mathbf{s}_i)$.

Proof. The Theorem follows from Lemma 5.1 for $E = (v_i; i \in \omega)$ an enumeration of M with $E_l = A$ and $E_m = B$. \square

Corollary 5.1. *Let $\mathbf{t} \subseteq \mathbf{s}$ be two Katětov functions of M with $\text{rank}(\mathbf{t}) = \text{rank}(\mathbf{s})$.*

Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in \text{dom}(\mathbf{t})$ and $\alpha(x) \in \text{orb}(\mathbf{s})$ for all $x \in \text{orb}(\mathbf{t})$, that is with $\text{orb}_{\alpha(M)}(\mathbf{t}) \subseteq \text{orb}_M(\mathbf{s})$.

6 The first block of \mathcal{D} and orbit distances

Let \mathcal{D} be a universal set of numbers and \mathcal{B} the first block of \mathcal{D} , that is $1 := \min \mathcal{B} = \min(\mathcal{D} \setminus \{0\})$. Let $M = (M, d) \in \mathcal{U}_{\mathcal{D}}$.

Lemma 6.1. *Let A be a finite subset of M and $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ three Katětov function with $\text{dom}(\mathbf{p}_i) = A$ for $i \in 3$. Then the triangle $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ with distance function d_{\min} is metric.*

Proof. Assume for a contradiction that

$$d_{\min}(\mathbf{p}_0, \mathbf{p}_1) + d_{\min}(\mathbf{p}_0, \mathbf{p}_2) < d_{\min}(\mathbf{p}_1, \mathbf{p}_2).$$

Let $w_0 \in \text{orb}(\mathbf{p}_0)$. There exist points $w_1 \in \text{orb}(\mathbf{p}_1)$ and $w_2 \in \text{orb}(\mathbf{p}_2)$ with $d(w_0, w_1) = d_{\min}(\mathbf{p}_0, \mathbf{p}_1)$ and $d(w_0, w_2) = d_{\min}(\mathbf{p}_0, \mathbf{p}_2)$. Then

$$\begin{aligned} d(w_0, w_1) + d(w_0, w_2) &\geq d(w_1, w_2) \geq d_{\min}(\mathbf{p}_1, \mathbf{p}_2) > \\ d_{\min}(\mathbf{p}_0, \mathbf{p}_1) + d_{\min}(\mathbf{p}_0, \mathbf{p}_2) &= d(w_0, w_1) + d(w_0, w_2). \end{aligned}$$

\square

Definition 6.1. *Let A be a finite subset of M and $s \in \omega$ and for every $i \in s$ let \mathbf{p}_i be a Katětov function with $\text{dom}(\mathbf{p}_i) = A$. Let $r \in \mathcal{B}$ with $1 \leq r^{(-)} < r$. Then, the \mathcal{D} -graph with set of points $\{0, 1, 2, \dots, s-1\} = s$ and distance function d' so that for all $i, j \in s$:*

$$d'(i, j) = \begin{cases} 0 & \text{if } i = j; \\ d_{\min}(\mathbf{p}_i, \mathbf{p}_j), & \text{if } d_{\min}(\mathbf{p}_i, \mathbf{p}_j) \geq r; \\ \in \{r^{(-)}, r\}, & \text{otherwise,} \end{cases}$$

is an r -levelling distance function on the indices of $(\mathbf{p}_i; i \in s)$.

Lemma 6.2. *Let A be a finite subset of M and $s \in \omega$ and for every $i \in s$ let \mathbf{p}_i be a Katětov function with $\text{dom}(\mathbf{p}_i) = A$. Then the \mathcal{D} -graph with set of points $\{\mathbf{p}_i : i \in s\}$ and distance function d_{\min} is metric.*

Let $r \in \mathcal{B}$ with $1 < r$. Then every r -levelling distance function d' on the set s of indices of $(\mathbf{p}_i; i \in s)$ is a metric space so that for $i \neq k \neq j$:

$$\begin{aligned} |d'(k, i) - d'(k, j)| &\leq d_{\min}(\mathbf{p}_i, \mathbf{p}_j) \text{ if } \mathbf{p}_i \neq \mathbf{p}_j \text{ and} \\ |d'(k, i) - d'(k, j)| &\leq 1 \text{ if } \mathbf{p}_i = \mathbf{p}_j. \end{aligned}$$

Proof. It follows from Lemma 6.1 that the \mathcal{D} -graph with set of points $\{\mathbf{p}_i : i \in s\}$ and distance function d_{\min} is a metric space.

The \mathcal{D} -graph with set of points s and distance function d' is a metric space because all triangles i, j, k are metric in the distance function d' as can be easily verified.

Let $\mathbf{p}_i = \mathbf{p}_j$. Then $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) = d_{\min}(\mathbf{p}_k, \mathbf{p}_j)$. If $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) \geq r$ then $d'(k, i) = d'(k, j)$ and hence $|d'(k, i) - d'(k, j)| = 0 < 1$. If $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) < r$ then $|d'(k, i) - d'(k, j)| \leq |r - r^{(-)}| \leq 1$ according to Theorem 3.1.

Let $\mathbf{p}_i \neq \mathbf{p}_j$. If both $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) < r$ and $d_{\min}(\mathbf{p}_k, \mathbf{p}_j) < r$ then $|d'(k, i) - d'(k, j)| \leq |r - r^{(-)}| \leq 1 \leq d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$. If $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) \geq r$ and $d_{\min}(\mathbf{p}_k, \mathbf{p}_j) < r$ then $|d'(k, i) - d'(k, j)| \leq |d_{\min}(\mathbf{p}_k, \mathbf{p}_i) - r^{(-)}| \leq |d_{\min}(\mathbf{p}_k, \mathbf{p}_i) - d_{\min}(\mathbf{p}_k, \mathbf{p}_j)| \leq d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$; the last inequality following from Lemma 6.1.

If both $d_{\min}(\mathbf{p}_k, \mathbf{p}_i) \geq r$ and $d_{\min}(\mathbf{p}_k, \mathbf{p}_j) \geq r$ then $|d'(k, i) - d'(k, j)| = |d_{\min}(\mathbf{p}_k, \mathbf{p}_i) - d_{\min}(\mathbf{p}_k, \mathbf{p}_j)| \leq d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$. \square

Corollary 6.1. *Let A be a finite subset of M and $s \in \omega$ and for every $i \in s+1$ let \mathbf{p}_i be a Katětov function with $\text{dom}(\mathbf{p}_i) = A$ and $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$ and $i, j \in s$. Let $t \in s$ and $\mathbf{p}_t = \mathbf{p}_s$ and $v \in \text{orb}(\mathbf{p}_s)$. Let $1 \leq r^{(-)} < r \in \mathcal{B}$ and $\text{rank}(\mathbf{p}_i) \in \{r, r^{(-)}\}$ for all $1 \leq i \in s+1$.*

Then for every r -levelling distance function d' on the indices of $(\mathbf{p}_i; i \in s+1)$ there exists a set $\{w_i \in \text{orb}(\mathbf{p}_i) : i \in s+1\}$ of points with $w_s = v$ and $d(w_i, w_j) = d'(i, j)$. Also:

$$|d(w_k, w_i) - d(w_k, w_j)| \leq d_{\min}(\mathbf{p}_i, \mathbf{p}_j) \text{ for } i, j, k \in s \text{ with } i \neq j \neq k \neq i.$$

Proof. If $d_{\min}(\mathbf{p}_i, \mathbf{p}_j) \geq r$ then $d'(i, j) = d_{\min}(\mathbf{p}_i, \mathbf{p}_j) \in d(\mathbf{p}_i, \mathbf{p}_j)$. If $d_{\min}(\mathbf{p}_i, \mathbf{p}_j) < r$, it follows from $\text{rank}(\mathbf{p}_i) \in \{r^{(-)}, r\}$ or $\text{rank}(\mathbf{p}_j) \in \{r^{(-)}, r\}$ according to Corollary 4.1 Item 3. that $d_{\max}(\mathbf{p}_i, \mathbf{p}_j) \geq r$ and therefore $d'(i, j) \in d(\mathbf{p}_i, \mathbf{p}_j)$. Hence the Corollary follows from Lemma 6.2 and Corollary 4.2. \square

7 The central extension theorem

Let \mathcal{D} be a universal set of numbers and \mathcal{B} the first block of \mathcal{D} , that is $1 := \min \mathcal{B} = \min(\mathcal{D} \setminus \{0\})$. Let $M = (M, d) \in \mathcal{U}_{\mathcal{D}}$.

Definition 7.1. *Let $S \subseteq M$ and $1 < r \in \mathcal{B}$. A Katětov function \mathbf{p} of M with $\text{rank}(\mathbf{p}) = r$ is extendible into S on M if for every copy $H = (H; d)$ of M in M with $\text{dom}(\mathbf{p}) \subseteq H$ and every Katětov function \mathbf{p}' with $\text{dom}(\mathbf{p}') \subseteq H$ and $\mathbf{p} \subseteq \mathbf{p}'$*

and $\text{rank}(\mathbf{p}') = r$ there exists an embedding α of H into H with $\alpha(x) = x$ for all $x \in \text{dom}(\mathbf{p}')$ and a Katětov function \mathbf{g} with $\text{dom}(\mathbf{g}) \subseteq \alpha(H)$ and with $\mathbf{p}' \subseteq \mathbf{g}$ and $\text{rank}(\mathbf{g}) = r^{(-)}$ so that $\text{orb}_{\alpha(H)}(\mathbf{g}) \subseteq S$.

Note that if a Katětov function \mathbf{p} with $\text{rank}(\mathbf{p}) = r$ is extendible into S on M and if H is a copy of M in M and \mathbf{q} is a Katětov function of H with $\mathbf{p} \subseteq \mathbf{q}$ and $\text{dom}(\mathbf{q}) \subseteq H$ and $\text{rank}(\mathbf{q}) = r$, then \mathbf{q} is extendible into S on H .

Lemma 7.1. *Let $1 \leq r^{(-)} < r \in \mathcal{B}$ and let A be a finite subset of M and $S \subseteq M$ and $t \in s \in \omega$ and \mathbf{p}_i a Katětov function for every $i \in s$ so that:*

1. $\text{dom}(\mathbf{p}_i) = A$ for all $i \in s$ and $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$.
2. $\text{rank}(\mathbf{p}_i) = r$ for $1 \leq i \in t+1$ and $\text{rank}(\mathbf{p}_i) = r^{(-)}$ for $t < i \in s$.
3. \mathbf{p}_i for $1 \leq i \leq t$ is extendible into S and if $t = 0$ then $\text{rank}(\mathbf{p}_0) = r$ and \mathbf{p}_0 is extendible into S .

Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in A$ and with image $H = (H; d)$ and a point $v \in H$ and for every $i \in s$ a Katětov function \mathbf{p}'_i so that:

1. $\text{dom}(\mathbf{p}'_i) = A \cup \{v\}$ and $\mathbf{p}_i \subseteq \mathbf{p}'_i$ for all $i \in s$ and $\mathbf{p}'_i \neq \mathbf{p}'_j$ for $i \neq j$.
2. $\text{rank}(\mathbf{p}'_i) = \text{rank}(\mathbf{p}_i)$ for all $1 \leq i \in s$ with $i \neq t$ and $\text{rank}(\mathbf{p}'_t) = r^{(-)}$.
3. \mathbf{p}'_i for $1 \leq i < t$ is extendible into S and if \mathbf{p}_0 is extendible into S and $t \geq 1$ then \mathbf{p}'_0 is extendible into S .
4. $\text{orb}_H(\mathbf{p}'_t) \subseteq S$.
5. $d_{\min}(\mathbf{p}'_i, \mathbf{p}'_j) = d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$ for all $i, j \in s$ with $i \neq j$.

Proof. Because \mathbf{p}_t is extendible into S there exists an embedding α' of M into M with $\alpha'(x) = x$ for all $x \in A$ with image $M' = (M', d)$ and a Katětov function \mathbf{g} with $\mathbf{p}_t \subseteq \mathbf{g}$ and $\text{rank}(\mathbf{g}) = r^{(-)}$ and $\text{orb}_{M'}(\mathbf{g}) \subseteq S$.

Let $v \in \text{orb}_{M'}(\mathbf{g})$ and \mathbf{g}'' the Katětov function with $\text{dom}(\mathbf{g}'') = \text{dom}(\mathbf{g}) \cup \{v\}$ and $\mathbf{g} \subseteq \mathbf{g}''$ and $\mathbf{g}''(v) = r^{(-)}$. It follows from Theorem 2.2 that \mathbf{g}'' is a Katětov function. Let \mathbf{g}' be the Katětov function with $\text{dom}(\mathbf{g}') = A \cup \{v\}$ and $\mathbf{g}' \subseteq \mathbf{g}''$. Then $\text{rank}(\mathbf{g}'') = \text{rank}(\mathbf{g}') = r^{(-)}$. It follows from Corollary 5.1 with \mathbf{g}' for t and with \mathbf{g}'' for s that there exists an isometry α'' of M' onto M' with $\alpha''(x) = x$ for all $x \in A \cup \{v\}$ and $\alpha''(x) \in \text{orb}_{M'}(\mathbf{g}'') \subseteq \text{orb}_{M'}(\mathbf{g})$ for all $x \in \text{orb}_{M'}(\mathbf{g}')$. Let $H = (H; d)$ be the image of α'' and $\alpha = \alpha'' \circ \alpha'$. Note that $\text{orb}_H(\mathbf{g}') \subseteq \text{orb}_{M'}(\mathbf{g}) \subseteq S$ and that $v \in H$.

Let \mathbf{p}_s be the Katětov function with $\mathbf{p}_s = \mathbf{p}_t$ and let d' be the r -levelling metric on $s+1$ given by:

1. $d'(s, t) = r^{(-)}$.
2. $d'(i, j) = \max\{r, d_{\min}(\mathbf{p}_i, \mathbf{p}_j)\}$ for $i, j \in s+1$ with $i \neq j$ and $(\{i, j\} \neq \{s, t\})$.

Corollary 6.1 provides a set of points $\{w_i \in \text{orb}(\mathbf{p}_i) : i \in s+1\}$ with $w_s = v$ and $d(w_i, w_j) = d'(i, j)$ for all $i, j \in s+1$. Let \mathbf{p}'_i for $i \in s$ be the type function with $\text{dom}(\mathbf{p}'_i) = A \cup \{w_s\}$ and $\mathbf{p}_i \subseteq \mathbf{p}'_i$ and $w_i \in \text{orb}(\mathbf{p}_i)$.

In order to see that \mathbf{p}'_i is a Katětov function we have to check the triangles of the form x, w_s, \mathbf{p}'_i with $x \in A$, which indeed are metric because they are isometric to the triangles x, w_s, w_i , which are substructures of M and hence metric. The Katětov functions \mathbf{p}_i are extendible into S for all $1 \leq i \in s$ because of the hereditary nature of the notion of being extendible into S .

Also $\mathbf{p}'_i(w_s) \geq r \geq \text{rank}(\mathbf{p}_i)$ for all $i \in s$ with $i \neq t$ and $\mathbf{p}_i \subseteq \mathbf{p}'_i$ implying that $\text{rank}(\mathbf{p}'_i) = \text{rank}(\mathbf{p}_i)$ for all $1 \leq i \in s$ with $i \neq t$ and if $\text{rank}(\mathbf{p}_0) = r$ and $t \neq 0$ then $\text{rank}(\mathbf{p}'_0) = \text{rank}(\mathbf{p}_0) = r$. Furthermore $\text{rank}(\mathbf{p}'_t) = r^{(-)}$ because $\mathbf{p}_t \subseteq \mathbf{p}'_t$ and $\text{dom}(\mathbf{p}'_t) = \text{dom}(\mathbf{p}_t) \cup \{w_s\}$ and $\text{rank}(\mathbf{p}_t) = r$ and $\mathbf{p}'_t(w_s) = r^{(-)}$. Because $\mathbf{p}'_t = \mathbf{g}'$ we have $\text{orb}_H(\mathbf{p}'_t) \subseteq S$. Then, according to the definitions of d_{\min} and d_{\max} and according to Corollary 6.1:

$$d_{\min}(\mathbf{p}'_i, \mathbf{p}'_j) = \max\{d_{\min}(\mathbf{p}_i, \mathbf{p}_j), |d(w_s, w_i) - d(w_s, w_j)|\} = d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$$

□

Corollary 7.1. *Let $1 \leq r^{(-)} < r \in \mathcal{B}$ and let A be a finite subset of M and $S \subseteq M$ and $s \in \omega$ and \mathbf{p}_i a Katětov function for every $i \in s$ so that:*

1. $\text{dom}(\mathbf{p}_i) = A$ for all $i \in s$ and $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$.
2. $\text{rank}(\mathbf{p}_i) = r$ for all $1 \leq i \in s$.
3. \mathbf{p}_i is extendible into S for all $1 \leq i \in s$.

Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in A$ and with image $H = (H; d)$ and a finite set $B \subseteq H$ with $A \cap B = \emptyset$ and for every $i \in s$ a Katětov function \mathbf{p}'_i so that:

1. $\text{dom}(\mathbf{p}'_i) = A \cup B$ and $\mathbf{p}_i \subseteq \mathbf{p}'_i$ for all $i \in s$ and $\mathbf{p}'_i \neq \mathbf{p}'_j$ for $i \neq j$.
2. $\text{rank}(\mathbf{p}'_i) = r^{(-)}$ for all $1 \leq i \in s$.
3. $\text{orb}_H(\mathbf{p}'_i) \subseteq S$ for all $1 \leq i \in s$.
4. $d_{\min}(\mathbf{p}'_i, \mathbf{p}'_j) = d_{\min}(\mathbf{p}_i, \mathbf{p}_j)$ for all $i, j \in s$ with $i \neq j$.
5. *If $\text{rank}(\mathbf{p}_0) = r$ and \mathbf{p}_0 is extendible into S then $B \subseteq H$ with $A \cap B = \emptyset$ can be chosen such that in addition to Items 1. to 4. above, also:*
 - 1'. $\text{rank}(\mathbf{p}'_0) = r - 1$.
 - 2'. $\text{orb}_H(\mathbf{p}'_0) \subseteq S$.

Proof. Follows by induction on $s - t$ from Lemma 7.1. □

Note: Let A be a finite subset of M and $S \subseteq M$ and \mathfrak{k} a Katětov function with $\text{dom}(\mathfrak{k}) = A$ and let $u \in \text{orb}(\mathfrak{k})$. Let \mathfrak{p} be a Katětov function with $\text{dom}(\mathfrak{p}) = A$ and $\text{rank}(\mathfrak{p}) = r \in \mathcal{B}$. Then, there exists a Katětov function \mathfrak{q} with $\text{dom}(\mathfrak{q}) = A \cup \{u\}$ and $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q}(u) = l \leq r^{(-)}$ if and only if $l \in d(\mathfrak{k}, \mathfrak{p})$ and $l \leq r^{(-)}$ if and only if $d_{\min}(\mathfrak{k}, \mathfrak{p}) \leq l \leq r^{(-)}$ and $l \in \mathcal{B}$. ($d_{\max}(\mathfrak{k}, \mathfrak{p}) \geq r^{(+)}$ if $r < \max \mathcal{B}$ according to Corollary 4.1 Item 3. and if $r = \max \mathcal{B}$ then $d_{\max}(\mathfrak{k}, \mathfrak{p}) \geq r$ according to Corollary 4.1 Item 2.)

Lemma 7.2. *Let A be a finite subset of M and $S \subseteq M$ and \mathfrak{k} a Katětov function with $\text{dom}(\mathfrak{k}) = A$. Let $v \in \text{orb}(\mathfrak{k})$ and let \mathfrak{P} be a set of Katětov functions \mathfrak{p} which are extendible into S and with $\text{dom}(\mathfrak{p}) = A$ and $\text{rank}(\mathfrak{p}) = r \in \mathcal{B}$ for which there exists a Katětov function \mathfrak{q} with $\text{dom}(\mathfrak{q}) = A \cup \{v\}$ and $\text{rank}(\mathfrak{q}) < r$, that is a set of Katětov functions \mathfrak{p} with $d_{\min}(\mathfrak{k}, \mathfrak{p}) < r$.*

Then there exists an embedding γ of M into M with $\gamma(x) = x$ for all $x \in A$ and a point $u \in \text{orb}_{\gamma(M)}(\mathfrak{k})$ so that $\text{orb}_{\gamma(M)}(\mathfrak{q}) \subseteq S$ for every Katětov function \mathfrak{q} with $\text{dom}(\mathfrak{q}) = A \cup \{u\}$ and $\mathfrak{q}(u) < r$ for which there is a Katětov function $\mathfrak{p} \in \mathfrak{P}$ with $\mathfrak{p} \subseteq \mathfrak{q}$. If $\mathfrak{k} \in \mathfrak{P}$ then $u \in S$.

Proof. If $\mathfrak{k} \in \mathfrak{P}$ let $s = |\mathfrak{P}|$ and $(\mathfrak{p}_i; i \in s)$ an enumeration of \mathfrak{P} with $\mathfrak{k} = \mathfrak{p}_0$. If $\mathfrak{k} \notin \mathfrak{P}$ let $s = |\mathfrak{P}| + 1$ and $(\mathfrak{p}_i; i \in s)$ an enumeration of $\mathfrak{P} \cup \{\mathfrak{k}\}$ with $\mathfrak{k} = \mathfrak{p}_0$. Then $(\mathfrak{p}_i; i \in s)$ satisfies the conditions of Corollary 7.1, which then supplies an embedding α with image $H = (H, d)$ and a set B and for every $i \in s$ a Katětov function \mathfrak{p}'_i . Let $u \in \text{orb}_H(\mathfrak{p}_0)$. If $\mathfrak{k} \in \mathfrak{P}$ then $\mathfrak{k} = \mathfrak{p}_0$ is extendible into S and hence $\text{orb}(\mathfrak{k} \cap S) \neq \emptyset$. In this case let $u \in \text{orb}(\mathfrak{k} \cap S)$.

For $i \in s$ let $f(i)$ be the set of all $l \in \mathcal{B}$ with $d_{\min}(\mathfrak{p}_i, \mathfrak{p}_0) \leq l \leq r^{(-)}$. Then, for every $l \in f(i)$, according to Corollary 7.1: $d_{\min}(\mathfrak{p}'_i, \mathfrak{p}'_0) = d_{\min}(\mathfrak{p}_i, \mathfrak{p}_0) \leq l \leq r^{(-)}$ and hence there exists a Katětov function $\mathfrak{s}_{i,l}$ with $\text{dom}(\mathfrak{s}_{i,l}) = A \cup B \cup \{u\}$ and $\mathfrak{p}'_i \subseteq \mathfrak{s}_{i,l}$ and $\mathfrak{s}_{i,l}(u) = l$ and $\text{rank}(\mathfrak{s}_{i,l}) = l$, because $\text{rank}(\mathfrak{p}'_i) = r^{(-)} \geq l$. It follows from $\mathfrak{p}'_i \subseteq \mathfrak{s}_{i,l}$ and $\text{orb}_H(\mathfrak{p}'_i) \subseteq S$ that $\text{orb}_H(\mathfrak{s}_{i,l}) \subseteq S$.

For every $i \in s$ and $l \in f(i)$ let $\mathfrak{t}_{i,l}$ be the Katětov function with $\mathfrak{t}_{i,l} \subseteq \mathfrak{s}_{i,l}$ and $\text{dom}(\mathfrak{t}_{i,l}) = A \cup \{u\}$. It follows from $\mathfrak{p}_i \subseteq \mathfrak{t}_{i,l}$ and $\text{rank}(\mathfrak{p}_i) = r$ that $\text{rank}(\mathfrak{t}_{i,l}) = l = \text{rank}(\mathfrak{s}_{i,l})$. Let $i, j \in s$ and $l \in f(i)$ and $m \in f(j)$ and $a \in \mathcal{B}$ minimal with $a \geq |l - m|$ and $b \in \mathcal{B}$ maximal with $b \leq l + m$. Note that $b \leq d_{\max}(\mathfrak{p}'_i, \mathfrak{p}'_j) \leq d_{\max}(\mathfrak{p}_i, \mathfrak{p}_j)$. Then $d(\mathfrak{t}_{i,l}, \mathfrak{t}_{j,m}) = d(\mathfrak{s}_{i,l}, \mathfrak{s}_{j,m})$, because:

$$\begin{aligned} d_{\min}(\mathfrak{t}_{i,l}, \mathfrak{t}_{j,m}) &= \\ &= \max\{d_{\min}(\mathfrak{p}_i, \mathfrak{p}_j), a\} = \max\{d_{\min}(\mathfrak{p}'_i, \mathfrak{p}'_j), a\} = d_{\min}(\mathfrak{s}_{i,l}, \mathfrak{s}_{j,m}) \end{aligned}$$

and

$$\begin{aligned} d_{\max}(\mathfrak{t}_{i,l}, \mathfrak{t}_{j,m}) &= \\ &= \min\{d_{\max}(\mathfrak{p}_i, \mathfrak{p}_j), b\} = b = \min\{d_{\max}(\mathfrak{p}'_i, \mathfrak{p}'_j), b\} = d_{\max}(\mathfrak{s}_{i,l}, \mathfrak{s}_{j,m}). \end{aligned}$$

Hence we can apply Theorem 5.1 to obtain an embedding β of H to H with $\beta(x) = x$ for all $x \in A \cup \{u\}$ with $\text{orb}_{\beta(H)}(\mathfrak{t}_{i,l}) \subseteq \text{orb}_H(\mathfrak{s}_{i,l}) \subseteq S$ for all $i \in s$ and $l \in f(i)$. Let $\gamma = \beta \circ \alpha$. \square

Theorem 7.1. *Let \mathfrak{q} be a Katětov function of $M = (M; d)$ with $\text{rank}(\mathfrak{q}) = r$ and $S \subseteq M$ so that \mathfrak{q} is extendible into S .*

Then there exists a copy $C = (C; d)$ of M in M with $\text{dom}(\mathfrak{q}) \subseteq C$ and $\text{orb}_C(\mathfrak{q}) \subseteq S$.

Proof. Let $H = (H; d)$ be a copy of M in M with $\text{dom}(\mathfrak{q}) \subseteq H$ and A a finite subset of H with $\text{dom}(\mathfrak{q}) \subseteq A$. Let \mathfrak{k} be a Katětov function of H with $\text{dom}(\mathfrak{k}) = A$. Let \mathfrak{P} be the set of Katětov functions \mathfrak{p} with $\mathfrak{q} \subseteq \mathfrak{p}$ and $\text{dom}(\mathfrak{p}) = A$ and $\text{rank}(\mathfrak{p}) = r$ and $d_{\min}(\mathfrak{p}, \mathfrak{k}) < r$. (Note that if $\mathfrak{q} \subseteq \mathfrak{k}$ and $\text{rank}(\mathfrak{k}) = r$ then $\mathfrak{k} \in \mathfrak{P}$.) Because \mathfrak{p} is extendible into S for every $\mathfrak{p} \in \mathfrak{P}$, there exists, according to Lemma 7.2, an embedding γ of H into H with $\gamma(x) = x$ for all $x \in A$ and a point $u \in \text{orb}_{\gamma(H)}(\mathfrak{k})$ so that $\text{orb}_{\gamma(H)}(\mathfrak{s}) \subseteq S$ for every Katětov function \mathfrak{s} with $\text{dom}(\mathfrak{s}) = A \cup \{u\}$ and $\mathfrak{s}(u) < r$ for which there is a Katětov function $\mathfrak{p} \in \mathfrak{P}$ with $\mathfrak{p} \subseteq \mathfrak{s}$. If $\mathfrak{k} \in \mathfrak{P}$ then $u \in S$.

It follows that the copy C can be constructed recursively. \square

8 Colouring $M_{\mathcal{D}}$

Let \mathcal{D} be a universal set of numbers with $\min(\mathcal{D} \setminus \{0\}) = 1$ and \mathcal{B} the first block of \mathcal{D} , that is $\min \mathcal{B} = 1$. Let $M_{\mathcal{D}} = (M_{\mathcal{D}}, d) \in \mathfrak{U}_{\mathcal{D}}$. Let $\chi : M \rightarrow \{0, 1\} = 2$ be a colouring of $M_{\mathcal{D}}$ and $S_0 := \{x \in M_{\mathcal{D}} : \chi(x) = 0\}$ and $S_1 := \{x \in M_{\mathcal{D}} : \chi(x) = 1\}$. Then χ induces a colouring of every copy M of $M_{\mathcal{D}}$ in $M_{\mathcal{D}}$.

Let $M = (M, d)$ be a copy of $M_{\mathcal{D}}$ in $M_{\mathcal{D}}$. For $E = (v_i; i \in \omega)$ an enumeration of M and α an embedding of M into M we denote by $\alpha(E)$ the enumeration $(\alpha(v_i); i \in \omega)$ of $\alpha(M)$ and for $n \in \omega$ by E_n the initial interval $(v_i; i \in n)$ of E .

A Katětov function \mathfrak{t} is *monochromatic in colour $i \in 2$* on M if $\chi(x) = i$ for all $x \in \text{orb}(\mathfrak{t})$ and \mathfrak{t} is *monochromatic* on M if there is $i \in 2$ so that \mathfrak{t} is monochromatic in colour i on M . For $r \in \mathcal{D}$, the enumeration E is *r -uniform* from $l \in \omega$ on if for every $n \in \omega$ with $l < n$ every Katětov function \mathfrak{p} with $\text{dom}(\mathfrak{p}) = E_n$ and $\text{rank}(\mathfrak{p}) = r$ is monochromatic.

For $r \in \mathcal{D}$ let $\mathbf{p}(r)$ be the statement:

$\mathbf{p}(r)$: For every copy $M = (M, d)$ of $M_{\mathcal{D}}$ in $M_{\mathcal{D}}$ and every enumeration $E = (v_i; i \in \omega)$ of M and $n \in \omega$ there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in E_n$ and a continuation of the enumeration of E_n to an enumeration $\alpha(E) = (\alpha(v_i); i \in \omega)$ of $\alpha(M)$ which is r -uniform from n .

Lemma 8.1. *Let $E = (v_i; i \in \omega)$ an enumeration of M . Let $n \in \omega$ and \mathfrak{p} a Katětov function with $\text{dom}(\mathfrak{p}) = E_n$ and $\text{rank}(\mathfrak{p}) = 1$. Then there exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in E_n$ so that \mathfrak{p} is monochromatic.*

Proof.

Case 1: There exists a Katětov function \mathfrak{s} with $\mathfrak{p} \subseteq \mathfrak{s}$ and $\chi(x) = 0$ for all $x \in \text{orb}(\mathfrak{s})$.

Then $\text{rank}(\mathfrak{s}) = \text{rank}(\mathfrak{p}) = 1$. According to Corollary 5.1 with \mathfrak{t} for \mathfrak{p} there exists an isometry α with $\alpha(x) = x$ for all $x \in \text{dom}(\mathfrak{p}) = E_n$ and $\alpha(x) \in \text{orb}(\mathfrak{s})$

for all $x \in \text{orb}(\mathbf{p})$. Then: $x \in \text{orb}_{\alpha(M)}(\mathbf{p})$ implies $x \in \text{orb}_M(\mathbf{s})$ and hence $\chi(x) = 0$.

Case 2: For every Katětov function \mathbf{s} with $\mathbf{p} \subseteq \mathbf{s}$ there exists a point $v \in \text{orb}(\mathbf{s})$ with $\chi(v) = 1$. Then we construct recursively an embedding α with $\alpha(x) = x$ for all $x \in E_n$ and so that \mathbf{p} is monochromatic in colour 1 on $\alpha(E)$. \square

Corollary 8.1. *Let $l \in \omega$. There exists an embedding α of M into M with $\alpha(x) = x$ for all $x \in E_l$ so that $\alpha(E)$ is 1-uniform from l . Hence $\mathbf{p}(1)$.*

Let \mathcal{D} be a homgen set and \mathcal{B} the block of \mathcal{D} with $\min(\mathcal{D} \setminus \{0\}) = \min \mathcal{B} := \mathbf{m}$ and $r \in \mathcal{B}$ with $r^{(-)} < r$. Let $M = (M, d)$ be a copy of $M_{\mathcal{D}}$.

Lemma 8.2. *Let $1 < r \in \mathcal{B}$ and $\mathbf{p}(r^{(-)})$. Let \mathbf{p} be a Katětov function of M with $\text{rank}(\mathbf{p}) = r$ which is not extendible into S_0 .*

Then there exists a copy $C = (C; d)$ of M in M with $\text{dom}(\mathbf{p}) \subseteq C$ and $\text{orb}_C(\mathbf{p}) \subseteq S_1$.

Proof. There exists a copy $H = (H; d)$ of M in M with $\text{dom}(\mathbf{p}) \subseteq H$ and a Katětov function \mathbf{p}' with $\text{dom}(\mathbf{p}') \subseteq H$ and $\mathbf{p} \subseteq \mathbf{p}'$ and $\text{rank}(\mathbf{p}) = r$ so that $\text{orb}_{\alpha(H)}(\mathbf{g}) \not\subseteq S_0$ for every embedding α of H into H with $\alpha(x) = x$ for all $x \in \text{dom}(\mathbf{p}')$ and all Katětov functions \mathbf{g} with $\text{dom}(\mathbf{g}) \subseteq \alpha(H)$ and $\mathbf{p}' \subseteq \mathbf{g}$ and $\text{rank}(\mathbf{g}) = r^{(-)}$. We will show that \mathbf{p}' is extendible into S_1 on H .

Let $L = (L; d)$ be a copy of H in H with $\text{dom}(\mathbf{p}') \subseteq L$ and γ an embedding with $\gamma(H) = L$ and $\gamma(x) = x$ for all $x \in \text{dom}(\mathbf{p}')$. Let \mathbf{p}'' be a Katětov function with $\text{dom}(\mathbf{p}'') \subseteq L$ and $\mathbf{p}' \subseteq \mathbf{p}''$ and $\text{rank}(\mathbf{p}'') = r$. Let $v \in \text{orb}_L(\mathbf{p}'')$ and $E = (v_i; i \in \omega)$ an enumeration of L so that $\text{dom}(\mathbf{p}'') = E_n$ and $v = v_n$ for $n = |\text{dom}(\mathbf{p}'')|$. Let \mathbf{g} be the Katětov function with $\text{dom}(\mathbf{g}) = E_{n+1}$ and $\mathbf{p}'' \subseteq \mathbf{g}$ and $\mathbf{g}(v_n) = r^{(-)}$. Because $\mathbf{p}(r^{(-)})$ there exists an embedding β of L into L with $\beta(x) = x$ for all $x \in E_n$ so that $\beta(E)$ is $r^{(-)}$ -uniform from n on. Then $\text{orb}_{\alpha(H)}(\mathbf{g}) \subseteq S_0$ or $\text{orb}_{\alpha(H)}(\mathbf{g}) \subseteq S_1$ for $\alpha = \beta \circ \gamma$. Also, $\alpha(x) = x$ for all $x \in \mathbf{p}'$ and $\text{dom}(\mathbf{g}) \subseteq \alpha(H)$ and $\mathbf{p}' \subseteq \mathbf{g}$ and $\text{rank}(\mathbf{g}) = r^{(-)}$. Hence $\text{orb}_{\alpha(H)}(\mathbf{g}) \subseteq S_1$.

It follows that \mathbf{p}' is extendible into S_1 on H and therefore from Theorem 7.1 that there is a copy $N = (N; d)$ of H in H with $\text{dom}(\mathbf{p}') \subseteq N$ and $\text{orb}_N(\mathbf{p}') \subseteq S_1$. According to Corollary 5.1 with \mathbf{p} for \mathbf{t} and \mathbf{p}' for \mathbf{s} and N for M , there exists an embedding δ of N into N with $\delta(x) = x$ for all $x \in \text{dom}(\mathbf{p})$ and $\delta(x) \in \text{orb}_N(\mathbf{p}') \subseteq S_1$. Let $C = \delta(N)$. \square

Lemma 8.3. *Let \mathbf{p} be a Katětov function of M with $\text{rank}(\mathbf{p}) = r \in \mathcal{B}$ and $2 \cdot r^{(-)} < r$ and with $\mathbf{p}(r^{(-)})$.*

Then there exists a copy $C = (C; d)$ of M in M with $\text{dom}(\mathbf{p}) \subseteq C$ and $\text{orb}_C(\mathbf{p}) \subseteq S_0$ or $\text{orb}_C(\mathbf{p}) \subseteq S_1$.

Proof. If \mathbf{p} is not extendible into S_0 the Lemma follows from Lemma 8.2. If \mathbf{p} is extendible into S_0 the Lemma follows from Theorem 7.1 \square

Corollary 8.2. *Let $r \in \mathcal{B}$ and $2 \cdot r^{(-)} < r$. (That is r is not the minimum of \mathcal{B} .) Then $\mathbf{p}(r^{(-)})$ implies $\mathbf{p}(r)$.*

Corollary 8.3. *Let \mathcal{D} be a universal set of numbers and \mathcal{B} the block of \mathcal{D} with $\min(\mathcal{D} \setminus \{0\}) = \min \mathcal{B}$ and let $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$ and $E = (v_i; i \in \omega)$ an enumeration of M . Then for every coloring $\chi : M \rightarrow 2$ there exists an embedding α of M into M so that for every $n \in \omega$ and every Katětov function \mathbf{p} with $\text{dom}(\mathbf{p}) = (\alpha(v_i); i \in n)$ and $\text{rank}(\mathbf{p}) = \max \mathcal{B}$ there exists $i \in 2$ with $\chi(x) = i$ for all $x \in \text{orb}(\mathbf{p})$.*

Note that for \mathcal{D} universal the class $\mathfrak{U}_{\mathcal{D}}$ is indivisible if an, and hence all, $M \in \mathfrak{U}_{\mathcal{D}}$ are indivisible.

Theorem 8.1. *Let \mathcal{D} universal consist of a single block. Then $\mathfrak{U}_{\mathcal{D}}$ is indivisible.*

Proof. Let $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$. It follows from Corollary 8.3 that there is a monochromatic Katětov function \mathbf{p} of M with $\text{rank}(\mathbf{p}) = \max \mathcal{D}$ and from Theorem 2.2 that $M \upharpoonright \text{orb}(\mathbf{p})$ is a copy of M because $\mathcal{D}_{\mathbf{p}} = \{n \in \mathcal{D} : n \leq 2 \cdot \max \mathcal{D}\} = \mathcal{D}$. \square

9 More than one euivalence class

Let \mathcal{D} be universal and \mathcal{B} be the block of \mathcal{D} with $\min(\mathcal{D} \setminus \{0\}) = \min \mathcal{B}$ and let $\max \mathcal{B} = \mathbf{m}$ and \mathcal{D} have at least two blocks. Let $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$ and let \sim denote the equivalence relation $\overset{\mathbf{m}}{\sim}$. For $x \in M$ denote the \sim -equivalence class containing x by $[x]$.

Lemma 9.1. *Let $A \in M/\sim$ and $a \in A$ and $X = \{x \in M : d(a, x) < \mathbf{m}\}$ and $C = M \setminus X$. Then $M \upharpoonright C$ is a copy of M in M .*

Proof. Let \mathbf{p} be a Katětov function of M with $\text{dom}(\mathbf{p}) \subseteq C$. According to Corollary 2.1 we have to show that $\text{orb}(\mathbf{p}) \cap C \neq \emptyset$. If $\text{rank}(\mathbf{p}) > \mathbf{m}$ Lemma ?? implies that there is a point $b \in \text{orb}(\mathbf{p})$ with $b \notin A$ and hence $\text{orb}(\mathbf{p}) \cap C \neq \emptyset$. Let $\text{rank}(\mathbf{p}) \leq \mathbf{m}$. Let $c \in \text{dom}(\mathbf{p})$ with $\mathbf{p}(c) \leq \mathbf{m}$. Then $\text{orb}(\mathbf{p}) \subseteq [c]$. Hence if $c \notin A$ then $\text{orb}(\mathbf{p}) \subseteq C$. Let $c \in A$. It follows that $\text{orb}(\mathbf{p}) \subseteq A$ and that $\mathbf{p}(x) \leq \mathbf{m}$ for all $x \in \text{dom}(\mathbf{p}) \cap A$ and that $\mathbf{p}(x) \geq 2 \cdot \mathbf{m}$ for all $x \in \text{dom}(\mathbf{p}) \setminus A$.

Let $b \in \text{orb}(\mathbf{p})$ and \mathbf{t} the Katětov function with $\text{dom}(\mathbf{t}) = \text{dom}(\mathbf{p}) \cup \{a\}$ and $\mathbf{p} \subseteq \mathbf{t}$ and $b \in \text{orb}(\mathbf{t})$. Then $\mathbf{t}(a) \leq \mathbf{m}$. Let \mathbf{q} be the type function with $\text{dom}(\mathbf{q}) = \text{dom}(\mathbf{p}) \cup \{a\}$ and $\mathbf{p} \subseteq \mathbf{q}$ and $\mathbf{q}(a) = \mathbf{m}$. In order to check that \mathbf{q} is a Katětov function we use inequality (1) from Section 2. Because \mathbf{p} is a Katětov function it remains to check that $|\mathbf{q}(a) - \mathbf{q}(x)| \leq d(a, x) \leq \mathbf{q}(a) + \mathbf{q}(x)$ for all $x \in \text{dom}(\mathbf{q})$. That is that:

$$|\mathbf{m} - \mathbf{p}(x)| \leq d(a, x) \leq \mathbf{m} + \mathbf{p}(x). \quad (2)$$

If $x \in A$ then $x \in C \cap A$ and $d(a, x) = \mathbf{m}$ and then Inequality (4) holds. If $x \notin A$ it follows from the fact that \mathbf{t} is a Katětov function and that $\mathbf{t}(x) = \mathbf{p}(x)$ and hence from Inequality (1) that:

$$|\mathbf{t}(a) - \mathbf{p}(x)| \leq d(a, x) \leq \mathbf{t}(a) + \mathbf{p}(x). \quad (3)$$

Because $\mathbf{p}(x) > \mathbf{m} \geq \mathbf{t}(a)$ Inequality (3) implies Inequality (2). \square

Let $E = (v_i; i \in \omega)$ be an enumeration of M and $E_n = (v_i; i \in n)$ for $n \in \omega$. The element $v_n \in M$ is *initial* if $[v_n] \cap E_n = \emptyset$. For v_n an initial point let \mathfrak{e}'_n be the Katětov function with $\text{dom}(\mathfrak{e}'_n) = E_n$ and $v_n \in \text{orb}(\mathfrak{e}'_n)$. Let \mathfrak{e}_n be the Katětov function with $\text{dom}(\mathfrak{e}_n) = E_{n+1}$ and $\mathfrak{e}'_n \subseteq \mathfrak{e}_n$ and $\mathfrak{e}_n(v) = \mathbf{m}$. (Because v_n is initial the rank of \mathfrak{e}' is larger than \mathbf{m} and hence it follows from Corollary 2.2 that \mathfrak{e}_n is indeed a Katětov function.) Then $\text{rank}(\mathfrak{e}_n) = \mathbf{m}$.

Let $\chi : M \rightarrow 2$ be a colouring of M . Then we obtain from Corollary 8.3 that there exists an embedding α of M into M so that for every initial point v_n the Katětov function $\alpha(\mathfrak{e}_n)$ is monochromatic. That is in the equivalence class $[\alpha(v_n)]$ the set of points of distance \mathbf{m} from $\alpha(v_n)$ is monochromatic. It follows then from Lemma 9.1 that by removing the points in $\alpha(M)$ at distance to v_n less than \mathbf{m} we obtain a copy of $H = (H, d)$ of $\alpha(M)$ in $\alpha(M)$ in which the \sim -equivalence class $[v_n] \cap H$ of H is monochromatic. Hence we obtained:

Lemma 9.2. *Let \mathcal{D} be universal and \mathcal{B} be the block of \mathcal{D} with $\min(\mathcal{D} \setminus \{0\}) = \min \mathcal{B}$ and let $\max \mathcal{B} = \mathbf{m}$ and \mathcal{D} have at least two blocks. Let $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$ and let \sim denote the equivalence relation $\overset{\mathbf{m}}{\sim}$. Let $\chi : M \rightarrow 2$ be a colouring of M . Then there exists a copy of M in M in which every \sim -equivalence class is monochromatic.*

For $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$ let M/\sim be the metric space with M/\sim as set of points and distance function d_{\min} with $d_{\min}(A, B) = \min\{d(x, y) : x \in A, y \in B\}$, see Lemma 3.3. Let \mathcal{D}_{\min} be the set of distances in M/\sim .

Lemma 9.3. *If \mathcal{D} is a universal set of numbers then \mathcal{D}_{\min} is universal and for $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$ the metric space $M/\sim = (M/\sim, d_{\min}) \in \mathfrak{U}_{\mathcal{D}_{\min}}$. If $N = (N; d_{\min})$ is a copy of M/\sim in M/\sim then $\bigcup N$ induces a copy of M in M .*

Proof. Let \mathfrak{p} be a Katětov function of M/\sim with $\mathfrak{p}(X) \in \mathcal{D}_{\min}$ for all $X \in \text{dom}(\mathfrak{p})$. According to Lemma 3.4 there exists an isometry $\beta : \text{dom}(\mathfrak{p}) \rightarrow M$ with $\beta(X) \in X$ for all $X \in \text{dom}(\mathfrak{p})$. Let \mathfrak{p}' be the Katětov function of M with $\text{dom}(\mathfrak{p}') = \{\beta(X) : X \in \text{dom}(\mathfrak{p})\}$ and $\mathfrak{p}'(\beta(X)) = \mathfrak{p}(X)$. Then \mathfrak{p}' is a Katětov function of M . Let $a \in \text{orb}(\mathfrak{p}')$ then $[a] \in \text{orb}(\mathfrak{p})$. Hence $M/\sim \in \mathfrak{U}_{\mathcal{D}_{\min}}$ according to Theorem 2.1.

Let \mathfrak{p} be a Katětov function of M with $\text{dom}(\mathfrak{p}) \subseteq \bigcup N$. Then \mathfrak{p} has a realization $a \in M$. There exists an isometry α of $(\{[x]_r \mid x \in \text{dom}(\mathfrak{p})\} \cup \{[a]_r\})$ into N which is the identity on $\{[x]_r \mid x \in \text{dom}(\mathfrak{p})\}$ and maps $[a]_r$ into N . Let $b \in \alpha([a]_r)$. Let \mathfrak{q} be the type function with $\mathfrak{p} \subseteq \mathfrak{q}$ and $\text{dom}(\mathfrak{q}) = \text{dom}(\mathfrak{p} \cup \{b\})$ and $\mathfrak{q}(b) = r$. Then \mathfrak{q} is a Katětov function and has a realization in $[a]_r$. \square

Theorem 9.1. *Let \mathcal{D} be a universal set of numbers and $M = (M, d) \in \mathfrak{U}_{\mathcal{D}}$. Then M is indivisible.*

Proof. Using Lemma 9.2 and Lemma 9.3 the Theorem follows by induction on the number of blocks in \mathcal{D} with Theorem 8.1 covering the case of a single block. \square

References

- [1] C. Delhommé, C. Laflamme, M. Pouzet, N. Sauer, *Divisibility of countable metric spaces*, European Journal of Combinatorics. Volume **28** Issue 6 (2007) 1746-1769.
- [2] R. Fraïssé, *Theory of relations*, Studies in Logic and the Foundations of Mathematics, 145, North-Holland Publishing Co., Amsterdam, 2000.
- [3] A. S. Kechris, V. Pestov and S. Todorcevic, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.* **15** (2005) 106-189.
- [4] J. Lopez-Abad and L. Nguyen Van Thé, The oscillation stability problem for the Urysohn sphere: A combinatorial approach, *Topology Appl.* **155** Issue 14 (2008) 1516-1530.
- [5] L. Nguyen Van Thé, N. Sauer, The Urysohn Sphere is oscillation stable, *Geometric and Functional Analysis* **19** Issue 2 (2009) 536-557.
- [6] L. Nguyen Van Thé, Structural Ramsey Theory of Metric Spaces and Topological Dynamics of Isometry Groups, *Memoirs of the American Mathematical Society*, 206, number 968, (2010).
- [7] E. Odell, T. Schlumprecht, The distortion problem, *Acta Mathematica* **173** (1994) 259-281.
- [8] V. Pestov, Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon. Revised edition of Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005]. University Lecture Series, 40. American Mathematical Society, Providence, RI, 2006.
- [9] W. Hodges, *Model Theory*, Cambridge Univ. Press, 1993.